Problem Solving

Introduction

An essential part of mathematics is finding patterns, posing questions about those patterns, and then solving the problems posed by those questions. As those problems are investigated, further interesting questions may naturally arise and lead to beautiful and intriguing developments for further investigation. Problem solving is at the heart of doing mathematics.

This paper looks at the process of doing problem solving. To distinguish problems from exercises, a problem will refer to a task whose method of solution is not readily apparent to the solver. Starting around 1980, Alan Schoenfeld created a framework for problem solving built on Polya's groundbreaking 1945 book *How to Solve It*. In his book *Mathematical Problem Solving*, Schoenfeld identifies four areas that make up a framework for problem solving: Resources (the collection of known math facts, procedures, skills), Problem-solving Strategies, Self-monitoring (watching over problem solving steps), and Beliefs (beliefs and attitudes about doing mathematics). Roughly speaking, each area of the framework attends to how the previous area is managed. This paper is made up of four chapters corresponding to Schoenfeld's four areas.

In the last few decades, various researchers have described variations on Schoenfeld's work. Notable among these frameworks is Singapore Math, which makes problem solving central to its curriculum and whose framework greatly resembles Schoenfeld's.

The process of solving a mathematical problem can be broken into phases as follows. The processes in these phases involve Strategies and Self-monitoring, which are described in detail in the appropriate chapters.

Phase 1: Understand the Problem Deeply.

Start by reading the problem very carefully, paying careful attention to what it is giving you and what it is asking for. Make sure that you have the necessary background in all the mathematical definitions and concepts involved in the problem – if not, this is the time to fill in any holes. Employ some basic strategies to get to know the problem: list out the givens and inferred givens, look for similar problems you have seen in the past, and find an effective way to represent the information of the problem.

Phase 2: Search for and Pursue Strategies

For difficult problems, this phase is where almost all your time and effort will be spent. This phase involves a cycle, perhaps repeated many times, of

> 1) Analyze the problem looking for entry points and places where the problem may be vulnerable to attack. Use this analysis to identify a problem-solving strategy to try.
> 2) Use the strategy identified in (1) as thoroughly as you can.

3) Constantly self-monitor to make sure pursuing this strategy continues to make sense. When it stops making sense, go on to step (4).
4) When you decide to suspend using the current strategy, you have a choice of how to cycle back to the start. You may decide that any one of the following is the appropriate next step:

a) You may think you are doing good things and are simply exhausted, in which case you will resume where you left off with this strategy when you have more energy (and who knows, your subconscious may have come up with some good ideas as you rested).
b) You may decide that this strategy is unlikely to be the best use of your resources right now (perhaps it is going nowhere or looks overly time consuming), in which case you should go back to (1) above and reanalyze the problem to find a new strategy to pursue.
c) You may suspect that you have misunderstood or missed something, in which case you should return to Phase 1 and possibly reach some new understandings.

Phase 3: Review your Solution.

At this point you believe you have a solution to the problem. Check that your solution is correct and that your steps and answers make sense. Make sure that your solution is presented in a clear manner that includes all the steps. Look to see if your result fits into other work you have done or see if it might lead to further interesting results.

Before diving into the rest of the paper, here are two enjoyable quotes about problem solving.

> (G. Polya) "Teaching to solve problems is education of the will. Solving problems which are not too easy, students learn to persevere through unsuccess, to appreciate small advances, to wait for the essential idea, to concentrate with all their might when it appears. If students have no opportunity in school to familiarize themselves with the varying emotions of the struggle for the solution, their mathematical education failed in the most vital point."

> (A. Schoenfeld) "Solving problems is the business of mathematicians; it is the excitement of mathematics. We owe it to those who will be the mathematicians of the future, to those who will use mathematics, and to those who would like a "feeling" for mathematics, to introduce them to the problem-solving experience. We hope and believe that the problem-solving approach to mathematics, throughout the curriculum and through a variety of problem courses, will convey to our students the excitement and beauty of mathematics. To the degree that we train our students to think independently and to <u>use</u> the knowledge at their disposal, we will have succeeded as teachers."

Chapter 1 - Resources

The term Resources will be used to refer to a student's mathematical knowledge base, an inventory of mathematical facts, procedures, and skills. It is a collection of schemas that a student has learned to recognize and knows how to use. For example, when an Algebra student sees the fraction 10/18, this triggers the recognition that there is a common factor of 2 that should be removed to obtain the equivalent fraction 5/9. Students have thousands of these schemas that they have been trained to recognize.

While the development of excellent Resources for a student is an important topic, there are only a few points I wish to make in this paper about them.

Many students overlook the distinction between being aware of a schema and having the ability to use a schema effortlessly. These students believe they have finished their work with a schema by knowing that a schema exists, and this belief creates false confidence. A schema should be practiced and mastered to the point of effortlessness so that it can be used without contributing to cognitive load, that is, so that it does not distract from the truly difficult parts of a problem. Everyone's short-term memory is limited. The more you need to use your short-term memory to use a schema that was not fully mastered, the less your short-term memory is available to work on the problem at hand.

Similarly, many students memorize schemas and simply know that they are true. Without a deeper understanding of a schema and its use, this knowledge is very brittle. Memorized material is not robust - it breaks and becomes useless when simple changes are made to a problem's circumstances. Also, memorization of facts and procedures is not only of dubious value, it is boring and uninteresting to do.

Students should continually be working on improving the state of their Resources. When going over homework or looking over a returned assessment, make an annotated list of schemas that need to be worked on. Perhaps bring this list to your teacher to go over together. If solving a quadratic equation using factoring is challenging for you, put it on the list. If you know how to solve linear equations, but you need more practice when the coefficients are fractions, put that on your list so that you will practice it to the point of effortless mastery. If you think the square root of a sum is the sum of the square roots, put that misconception on the list. Everyone has schemas that are partially understood, need more practice, or are even incorrectly understood - put those schemas on a list with an explanation of why they are there, and then systematically improve them!

Chapter 2 - Problem Solving Strategies

Problem solving strategies are general rules of thumb for solving problems. They are guidelines for what to do next when attacking an unfamiliar problem. No one strategy, or even all the strategies combined, can guarantee that progress will be made on a problem. Problem solving is an art with no automatic methods, and that is part of what makes it so much fun. Note that strategies are useless if the problem solver has not mastered the appropriate Resources for the problem domain.

Just as Resources are learned over time, so should Strategies. If a student learns Strategies a few at a time starting in elementary school and continuing on into Middle and Upper School, by the time college courses are reached, the student will have a wide array of problem-solving strategies mastered and available. Schoenfeld points out that a basic sounding strategy, such as “look at a simpler version of the problem,” is actually composed of dozens of more specific strategies more easily understood by students, such as “when looking at a problem involving general triangles, look at what happens for isosceles, right, or equilateral triangles.” Note that the distinction between Strategies and Resources can be subtle at times. The guideline that Strategies are useful rules of thumb for attacking problems will serve to keep us from being too fussy about when some very familiar or simple Strategy might be thought of as a Resource.

Singapore Math considers strategies an essential part of their approach to problem solving. They have a list of 13 strategies, with the first 11 to be mastered in the primary grades. Briefly, their list is: 1) Act out the problem; 2) Use a diagram or model; 3) Use guess-and-check; 4) Make a systematic list; 5) Look for patterns; 6) Work backwards; 7) Use before-after concept (work forward from givens and backward from goal); 8) Make suppositions (add or remove elements to or from the problem); 9) Restate the problem; 10) Simplify the problem; 11) Solve part of the problem (break it into subproblems); 12) Think of a related problem; 13) Use equations.

The strategies described in this chapter are categorized into levels. The first section consists of strategies to be employed at the start and end when solving any problem. After that section, Basic Strategies covers strategies that are useful up through about Algebra I. Intermediate Strategies are fairly general strategies useful in attacking more sophisticated problems. Advanced Strategies tend to be of narrower applicability and are often aimed at specific types of advanced problems. There is no distinct cutoff between the levels - there will be simpler problems that benefit from one of the advanced strategies, and there will be complex problems that benefit from the insights of basic strategies. Also note that some general strategy descriptions are repeated in the various levels; however, when this repetition occurs, the specific strategies associated with that category will be more advanced in the more advanced levels.

Strategies for the Start and End of Every Problem

No matter the level of difficulty of the problem, start every problem by learning deeply about it. It is essential to read the problem carefully with attention to detail and to look for possible points of confusion. The starting strategies listed in this section are meant to bring you to a deeper understanding of your problem. After you finish your solution to the problem, reflecting on your work and results is important for finding mistakes, improving your problem-solving skills, and discovering interesting new directions to investigate.

Start every problem by getting to know it deeply:

- Read the problem carefully and understand all the details.
- Recognize the nature of the material and make use of similar problems you have seen.
- Represent the information in a way that organizes it and makes the problem clearer.

End every problem by reflecting on your solution:

- Check your answer and your solution steps.
- Make sure your solution writeup is clear and easy to read.
- Reflect on the process of solving this problem and the results you obtained.

The following are detailed discussions of each of the strategies listed above.

■ Have you read the problem carefully and taken note of all the details? Be sure to read it several times. In their hurry to solve a problem, students often rush through reading a problem and skip over important details. It is surprising how often rereading a problem will produce reactions such as “oh, I get it now.”

Make sure you are being thorough by asking yourself the following questions.

What are you given and what do you know? Identify all facts, figures, and information and write them down. Try restating the problem in your own words to check your understanding.

Do you know and understand the definition of all the terms, concepts, and formulas used in the problem? If not, look them up and understand them before tackling the problem.

What do you assume to be true? Are these assumptions valid?

What new information can you infer directly from the givens? These are called inferred givens and should be written down along with the givens. For example, if you are told that there is a car that has traveled 100 miles in 2 hours, an inferred given is that the car has traveled at an average speed of 50 miles per hour.

What are the goals of the problem? What results are asked for in the problem?

Is there more than one question built into the problem? Be sure not to leave any out.

■ Have you seen a problem like this before? Are there problems you have looked at that have similar data, results, or unknowns? Does the nature of this problem suggest using methods to solve it that are like past problems?

The problems in our problem sets often build on earlier problems (perhaps several pages or days away). Taking a moment to recall related earlier problems can make the current problem much easier. Looking at how that earlier problem was solved (good notes help here) may give you a ready-made method for solving this problem.

Similarly, though less definitely, finding problems you have done with similar setups or results gives you problems that are useful in the same way.

Example You are asked to show that the only number that has all 1's that is a square is the number 1. You look at the result and think about what you know about numbers that are squares. You recall some earlier problems that numbers that are squares must be either of the form 3k or 3k+1, and they must also be of the form 4k or 4k+1. The first fact turns out not to be useful, but the second one solves the problem!

Example (From PEA Math 1) A ladder is leaning against a wall. Each time I step up on a rung I get 6 inches closer to the wall and 8 inches higher off the ground. The base of the ladder is 9 feet from the wall. How far up the wall does the ladder reach? Seeing this problem reminds you of problems where lines went up and over various amounts and other problems involving sides of triangles whose sides had fixed ratios. You look at some of the line problems and see the idea of using slopes and that helps you solve the problem. Or perhaps you look at the triangle problems and get the idea of using similar triangles and you use that to solve your problem.

What is the nature of this problem? Does the nature of this problem suggest taking certain approaches to solving it?

For example, if a problem looks similar to a quadratic equation, you may be able to transform it so that you can use your tools for solving quadratic equations.

Example (Polya) $a^4 – 5a^2 + 4 = 0$. This feels like it may be a quadratic equation in disguise. By setting $x = a^2$, this equation becomes $x^2 – 5x + 4 = 0$. Now you can use your factoring skills for quadratic equations to turn this into $(x – 4) (x – 1) = 0$, which gives the solutions x = 1 or 4. Finally, $x = a^2$ produces a = -1, 1, -2, or 2.

Add in the point that it is also important to find similar problems and then note minor differences that may be important for translating the earlier solution to this one – this may not be needed as it is mostly already here, but think about it.

■ Choose a representation of the problem that organizes the information or makes the problem logic clearer.

Have you tried drawing a picture, creating a diagram, or acting out the problem scenario?

Making the problem more concrete through visualization or acting can be a tremendous help in understanding the problem. Sometimes making sketches of the actions described in the problem and annotating the sketch with problem details can help a lot.

Example If 128 tennis players play a single-elimination tournament, playing no more than one match per day, how many days would it take and how many individual matches would occur? Look at a simpler version of this problem involving 8 players. You can act this out using 8 people standing at the front of the room, or you can act this out with pencil and paper by making a diagram of possible match outcomes.

Example Alice and Bob live 12 miles apart. Alice rides toward Bob at 14 miles an hour and Bob walks toward Alice at 4 miles an hour. How long does it take for them to meet in the middle? Drawing a picture of this may make it easier to see that they have 12 miles to cover at a combined speed of 18 miles an hour, so it will take them 2/3 of an hour.

Example How many ways can you stack three books on a table? If you try this with three books, you will see there are six ways to do it. If you think about your process, you find you had three choices for the bottom book, two choices for the second book, and one choice for the top book – this is what gives 3 * 2 * 1 = 6 ways to do it.

If the problem involves a fair amount of data, putting the data into a graph, table, bar chart, pie chart, or some other data representation may make it easier to understand the data and see patterns in it. Along similar lines, in the course of working on a problem, students will have examples and problem attempts scattered all over a page or several pages. Our brain is a fabulous recognizer of patterns, but it needs to have a chance to see the information clearly. Taking a minute to organize the data you have can make a result obvious or make it clearer what work remains to be done.

Drawing a diagram or graph is often a good idea, even when the problem seems algebraic in nature. Creating this new representation may give you a much better feel for the problem and may suggest new ideas to you. You might even solve the problem graphically.

Example You work a problem and produce the points (4, 3.5), (5, 5.5), and (7,9.5). If you make a table, you may see that the y-value increases twice as fast as the x-value. If you graph the points, it will be obvious that this data is linear with slope 2.

Example Tables are often an excellent way to organize information in a rate problem. Problem: It takes the first pump 6 hours to fill a tank and it takes the second pump 8 hours to fill the same tank. We want to use three pumps that will combine to fill the tank in 2 hours. How quickly must the third pump fill a tank? Once the table is set up, you see that the rates need to add up to 1/2 and the rest of the problem is easy

Pump	Time	Amount	Rate
First	6 hrs.	1 tank	1/6 tank/hr.
Second	8 hrs.	1 tank	1/8 tank/hr.
Third	x hrs.	1 tank	1/x tank/hr.

All Three	2 hrs.	1 tank	1/2 tank/hr.

Example Singapore Math makes extensive use of horizontal bar models to represent problems. The bars are used to illustrate how the parts of an entity can be put together to create a whole, or to see which parts are missing from the whole. When there are comparisons of several entities, their bar models can be layered to make comparing them easy. If you are interested in finding out more about this, please search online for the extensive materials available there.

Once a student becomes comfortable with algebraic methods, a standard form of representation is to use variables to translate the sentences of a problem into equations and inequalities.
Identify the things you will need to work with and give all, or most, of them names.
In particular, identify the things you need to solve for and give them names.
Translate the sentences into math – typically this is a collection of equations and inequalities.

Example Al's father is four times as old as Al is. In four years, Al's father will be three times as old as Al will be. How old are they. If you let A and F represent the current ages of Al and his father, respectively, the two sentences translate to F = 4A and (F + 4) = 3(A + 4). Substituting F = 4A into the second equation leads to 4A + 4 = 3A + 12, which is easily solved to discover A = 8 and F = 32.

You are done with the problem, now what? There are important ideas to consider once you have finished a problem. These ideas are often ignored due to time pressure or simply the feeling that the job is done, and it is time to move on to the next thing.

■ Have you checked your answer and your solution steps? While it is always a good idea to look back over your steps, if there is a mistake in those steps it is often easy to have the same thought processes while reviewing the steps as you did originally and thereby repeat the mistake and not catch it. One way to avoid this is to find alternate ways to do things. Here are some other ways of finding errors in your work.

a. Are the units or types of things in your answer sensible?

Example If your problem asks for the area of something and your answer is a length, then you have a mistake to find.

b. Are the amounts in your answer reasonable?

Example If your problem is calculating how many cars are needed to move some children and you end up with 5 2/3 cars, you have an unreasonable answer and need to reconsider how to calculate the correct answer.

c. Is your answer of reasonable size and quality? Is it absurdly large or small? Is it negative or 0 when it should be positive?

d. More generally, does your solution make intuitive sense now that you have worked your way to the end of the problem?

e. Try plugging your answer back into the problem and carry it through the original problem steps. Does your answer make all the steps work out correctly?
f. Is the correctness of each of your problem-solving steps clear to you? For any steps you are unsure of: can you redo it in a different way, is there an independent way to check it, can you identify the place of uncertainty and prove to yourself that it works?

■ Be sure to read through your solution to ensure that it would be easy for another person to follow and understand it – your exposition is important! Good ideas presented in an uncompelling way are wasted. Make sure that every important step is included and is clearly explained. If you find you cannot explain something, that may mean that it is incorrect or that you have further work to do.

■ Reflect on the process of solving this problem and the results you obtained.
a. If a result is new to you, record it for future use. Think about how this result fits into other similar results or results about similar material.
b. If the problem-solving process was new or particularly interesting, make a note of it so you will not need to reinvent it the next time you encounter a problem of this type.
c. If the problem solution had some unusual or notable applications of problem-solving strategies or uses of theorems, note those too.
d. If you made some mistakes, think about recording those and how they occurred. Making a systemic list of your mistakes will raise your awareness of the types of mistakes you tend to make and make it easier to reduce their frequency.
e. Reflect on how well you did with self-monitoring. Did you spend a large amount of time following an idea that was clearly never going to work, was doomed to take a huge amount of time, or that even if it was successful was unlikely to get you closer to your goal?

Basic Strategies

- Look at simpler versions or variations of the problem. Do examples and look for patterns.
- Break the problem into pieces easier to tackle independently.
- Work the problem forward and backward. Create intermediate goals.
- Try guess and check or estimating the answer.
- Work together when it is allowed. Describe to a listener your ideas about the problem.
- Use Prealgebra and Algebra I strategies.

The following are detailed discussions of each of the problem strategies listed above.

■ Have you looked at simpler versions of the problem? Along the same lines, have you tried lots of examples and looked for patterns in the way the examples work out? Sometimes, just the process of working the examples illuminates what the process of the problem is and how to solve it. It is surprising how powerful this simple idea is and how often it is neglected. Students are reluctant to start with the easiest examples, thinking them too trivial - given how quick and useful the easiest examples often are, they should be encouraged to start very simply.

Example If you travel at (x + 3.2) mph and want to travel 63.2 miles, how long will it take? A student may be daunted by the complicated expressions and be unsure how to proceed. However, if they realize that solving this problem uses the same steps as solving the simpler problem of finding the travel time for going 120 miles traveling at 30 mph, then they will know which steps to follow.

Example What is the highest power of 2 that evenly divides 1000! ? If the student has never thought about this, solving this problem can be overwhelming. However, if the student looks at solving the same problem for 2!, 3!, 4!, 5!, 6!, 7!, and 8!, in a very short time a pattern will emerge that can be used to solve the larger problem.

Example What is the ones digit of 7^{1000}? When students are asked for simpler versions of this problem, they will often start with 7^5 or even 7^{20}. You should usually start with the very simplest and smallest example when possible. Doing this problem for 7^1, 7^2, 7^3, 7^4, 7^5, and 7^6 will only take a minute or two, and the pattern for how to solve the problem will emerge and be available.

Example What is the product of all the divisors of a number? At first, this sounds quite complicated. However, if you write out the first examples of this for divisors of 4, 5, 6, 7, 8, and 9, you will quickly discover that this has a fairly simple formula.

Example In a basic form of Nim, two players start with a number and then they take turns subtracting 1 or 2 from the current number. The player who makes the current number 0 wins. What is the winning strategy? If the starting number is

50, this is an imposing problem. However, by starting with 1 and doing the first five or six examples, the pattern emerges, and it becomes an easy problem.

Have you looked for variations of this problem that may be easier to solve? Perhaps there are problems analogous to the original problem that you can solve and learn from. Similarly, can you think of ways of generalizing or specializing the original problem that would be easier to solve or interesting to investigate?

Similar to looking at easier versions of a problem, sometimes you can find a variation of the problem that is easier to solve. While solving this problem may not give you a method for solving the original problem, its line of reasoning may be similar enough that it gives you insights into how to solve your original problem. Along similar lines, changing a problem to something more general or specific can create a problem that is easier to solve.

Example (Polya) Suppose you are told there are two ships traveling in straight lines at constant speeds, and you are asked to calculate their distance apart when they are closest. At first, you are stumped and wonder how to solve it. You decide to consider a special case where one of the ships has speed 0 and is not moving. You have seen this before and know that the shortest distance is what the distance will be when the line between the two ships is perpendicular to the motion of the moving ship. You then realize that you can make the original problem into this more specialized problem by subtracting the motion of one of the ships from the motion of both of the ships!

When dealing with problems in which an integer parameter n plays a prominent role, it may be of use to examine values of n = 1, 2, 3, … in sequence, in search of a pattern.

If the problem in its current form is too difficult, relax one of the conditions. Ask for a little less than the current problem does, while making sure that the problem you consider is of the same nature. Now there should be more than one solution to the new problem. Look at the collection of solutions to the easier problem and see if the solution to the original is among them.

In general, if a problem is true about a general class of objects, look at what happens for the simplest members of that class. For example, if it involves triangles, look at isosceles, right, or equilateral triangles. For problems involving an arbitrary chord in a circle, look at the special case where the chord is a diameter.

Example There is a theorem that says that the size of an inscribed angle in a circle is half the size of the intercepted arc. To start to prove this, prove it first for angles that have a diameter as one of their sides. Proving this special case is relatively straightforward, and the result of the special case may be used as a basis for a general proof.

- Have you tried breaking the problem into pieces that are easier to tackle independently?

There are two ways that this can be helpful in problems. One is to establish a string of subgoals that when traversed will give the full solution. The other is to break the problem into independent cases each of which can be attacked more easily than the original problem – that option will be discussed in the Intermediate Strategies section. #### maybe include it here?

Creating subgoals: In your career as students, you have seen many problems like this where the problem writer has broken the solving of a harder problem into multiple steps for you, often labeled parts a, b, c, and d. As you become more sophisticated as a problem solver, you will be expected to find for yourself the places where those intermediate results can be obtained.

Example Write this ####

The creation of a subgoal often occurs without premeditation. You may find that you have successfully made it part of the way to the solution, but now you are temporarily stumped. That waypoint becomes a subgoal you have reached – with luck it will be part of a successful path to the full solution.

■ Have you tried working the problem forward and backward? Have you considered creating intermediate goals? Be sure to value and write down partial solutions!

Clean this up and write more.

While simpler problems can often be solved by going from beginning to end in a straight line with relatively little difficulty in between, more complex problems often call for more problem-solving flexibility and ingenuity. To find your way to completing the solution, you may need to set up intermediate goals to reach. In some circumstances, you may find it easier to work backward from goals rather than reaching those goals in a forward direction. Sometimes, part of your work will be forward and other parts will be backward – this often happens in more difficult geometry problems. During all this work, be sure to cherish your partial solutions – you may come back to your work after a while and suddenly see how to finish it, or you may show it to someone who may have just the idea you were missing.

Example (Polya) You have a 4-pint bucket and a 9-pint bucket. You are challenged to return from a lake holding exactly 6 pints of water. How do you do it? After some thought, you realize that at the end you must have 6 pints in the 9-pint bucket, which means you had to find a way to pour off 3 pints from the full bucket. To pour off 3 pints, you had to have 1 pint in the 4-pint bucket. To get 1 pint in the 4-pint bucket you can fill up the 9-pint bucket and pour 4 pints out of it twice. Working backwards from the desired result is far easier than experimenting with all the ways to start this problem and finding the one way of succeeding.

■ Be playful with the problem. Sometimes you should just mess around with a problem, enjoy the interplay of the mathematics, and for a moment not concern yourself as much with where things are leading. There are several ways to be playful.

One playful approach is to use educated guess and check to find a solution that works. Along those lines, try estimating what the solution should look like and see how close you can come. Students often feel that guessing is not solving the problem, that it lacks merit. This is not the case! Attacking a problem by making educated guesses has several benefits. First, it may provide you with the requested answer. Second, the process of deciding how to make those guesses combined with the collection of results obtained in this way teaches you a great deal about the internal structure of the problem.

Example You have 56 bicycles and tricycles in the shop, and you have a total of 138 wheels to pump up. How many tricycles do you have? If you guess the number of tricycles and guess 20, 25, and 30, you will end up with 132, 137, and 142 wheels. You will notice that the number of wheels goes up as the number of tricycles increases and quickly get the answer of 26 tricycles. You may also notice that the number of wheels increases by one with each additional tricycle and arrive at the equation 2*56 + tricycles = 138.

Another way of playing is to come up with several preliminary or partial ideas. Do not worry that you do not see how these ideas can be used. Record them so that the ideas are not forgotten, and then keep playing around with other new ideas.

Perhaps above all else, remain undaunted and be persistent. Keep playing with ideas for this problem and have fun with them. This may be a good moment to look at some of the thoughts in the Self-Monitoring chapter.

■ Work together when it is allowed. Along the same lines, describe to a listener all your ideas about your problem.

Doing mathematics need not be a lonely, solitary activity. Sharing ideas and problem-solving techniques can be fun and help all involved to grow and become better problem solvers by learning from each other. Avoid simply giving or getting answers when working with others. Give guiding hints or questions so that the other students have the pleasure of solving the puzzle; the questions from this document may help a lot with giving each other the appropriate kinds of help!

Telling someone your ideas about a problem would seem to be a waste of time. However, you will be surprised how often when you are in the middle of describing your ideas you suddenly see your way clear to the end of the problem (be sure to thank your listener).

■ Use Prealgebra and Algebra I strategies. This is a list of strategies to use for early Algebra problems.

This needs a bunch of work

Combine like terms if it seems to simplify the expression.

Simplify complex fraction expressions. Remove common factors from the numerator and denominator. Use a common denominator to combine fractions being added or subtracted. Use fraction rules to combine fractions being multiplied or divided.

For equations involving a single variable, move all terms involving it to one side and try to isolate the variable

Consider solving an equation graphically. If you graph both sides of an equation, the points of intersection of the two graphs will be solutions to the equation.

If you are solving an equation involving fraction coefficients, consider multiplying both sides of the equation by the least common multiple of all the denominators.

For equations of the form $(\text{something})^2 = a^2$, take the square root of both sides and be sure to consider both roots.

For degree two equations not of the form $(\text{something})^2 = a^2$, move all terms to one side and use your tools for solving quadratic equations.

If an expression looks like it is easy to factor, factor it and see if that simplifies solving the problem. The Difference of Squares pattern is fairly common and is easy to work with.

For equations of degree higher than one, put all the terms on one side and see if you can factor it.
For equations involving square roots (or nth roots), isolate the square root (nth root) to one side of the equation and square (nth power) both sides of the equation. Look out for introducing false solutions when you do this.
If you are adding or subtracting a fraction with a square root in the denominator, consider rationalizing the denominator of the fraction to make it easier to combine with the other fraction.

Intermediate Strategies

- Look at simpler versions of the problem. Try lots of examples and look for patterns.
- Break the problem into pieces easier to tackle independently.
- See if an inductive process would help.
- Identify auxiliary items you can add to the problem that may make it easier to solve.
- Work the problem backwards and forwards and see if they meet in the middle.
- Play with the problem. Write down and identify partial solutions and preliminary ideas.
- If you have trouble proving that something is true, look for counterexamples. #### combine this with next two?
- Assume the result or your proposed answer is true. What must be true for that to happen?
- Consider proof by contradiction or contrapositive - assume the conclusion is false.
- Exploit symmetries in the problem.
- Look for extreme points in the problem that may force actions or relationships.
- Use Geometry strategies.
- Use Algebra II and Precalculus strategies.
- Use Calculus strategies.

When you are done with the problem:

- Check your answer and your solution steps.
- Can you think of a different way to solve this problem?
- Think of ways to generalize or specialize the original problem that would be interesting. The problems given in your homework do not need to be the end of your math explorations!

The following are detailed discussions of each of the problem strategies listed above.

█ Have you looked at simpler versions of the problem? Along the same lines, have you tried lots of examples and looked for patterns in the way the examples work out? Sometimes, just the process of working the examples illuminates what the process of the problem is and how to solve it.
Yes, this is repeated from the Basics section. This strategy has more basic versions and some sophisticated ones, and it is worth looking at all of them.

Also, simplify the problem. Should this be separate or part of this one?
finish writing
Consider a similar problem with fewer variables.

Identify important variables, a variable to eliminate, keep 1 variable constant & observe changes

Recognize chunks and make substitutions

When dealing with geometric figures, look at special cases that have minimal complexity. For example, regular polygons; isosceles, right, or equilateral polygons; semi- or quarter circles instead of arbitrary sectors.

For geometric arguments, convenient values for computations can often be used WLOG (e.g. making the radius of a circle to be 1).

■ Have you tried breaking the problem into pieces that are easier to tackle independently?

There are two ways that this can be helpful in problems. One is to establish a string of subgoals that when traversed will give the full solution. The other is to break the problem into independent cases each of which can be attacked more easily than the original problem.

Example #### Write this

Creating cases:

finish writing – give explicit examples

■ See if an inductive process would help. Some problems have a natural way of building from one step to the next. If so, this may be of use in solving the problem. If there is an infinite process involved, consider using mathematical induction. Some problems are easier to show using induction for the given formula than by creating the formula from scratch.

Example (Schoenfeld) You are given real numbers a, b, c, d, and e, each of which lies between 0 and 1. Prove that $(1 - a)(1 - b)(1 - c)(1 - d)(1 - e) > 1 - a - b - c - d - e$. The simplest way to do this problem is to first show that $(1 - a)(1 - b) > 1 - a - b$ (which is fairly easy), then use that to show $(1 - a)(1 - b)(1 - c) > 1 - a - b - c$, and so on.

Example Show that every 2^n by 2^n checkerboard with an arbitrary square removed can be tiled using right triominoes. A right triomino is a tile consisting of three squares put together in an L shape. Attacking this problem directly is difficult due to all the different possibilities to address. Using induction, the problem is quite simple to solve.

Example Prove that the sum of the first n odd numbers is n^2. If you ignore the elegant geometric proof of this, the most natural way to proceed is to use mathematical induction and the proof is quite straightforward.

2) If there is an integer parameter, look for an inductive argument. Either list out f(1), f(2), etc and look for patterns; also sometimes there is an easy way to see how f(n) leads to f(n+1) which leads to a nice inductive approach.

1) If there is an integer parameter (n) in a problem, look at special cases n = 1, 2, 3, … Look for patterns and observe the calculations for an inductive mechanism.

■ Are there any auxiliary items you can add to the problem that may make it easier to solve?

The place this strategy most often comes up is in more difficult geometry problems. Adding an extra line segment, or even an extra portion of the diagram, can make the problem easier to solve; however, it can be tricky to see what those new items should be.

Example Prove that in triangle ABC, with congruent sides AB and AC, the angles at B and C are congruent. One way to prove this is to add the median from A to your diagram and use SSS on the two triangles just created.

■ Be willing to play with the problem and try things that may not look to pay off at first glance. Value your partial solutions and preliminary ideas. Also, write ideas down as they come to you so that you can look at them and be more fully aware of them.

Students may not realize the true importance of these. Upon further reflection on the problem, preliminary steps often become steppingstones to finishing a problem. After sleeping on the problem, an earlier partial solution may put you in a position to find your way to the end. Even if you do not finish the solution, someone may be able to take your partial solution and finish the problem – this also has the advantage that you can see what steps they used in that situation, and those steps may help you get unstuck in the future.

Play around with the problem and try different approaches even though you do not see any immediate use for the ideas. I am often surprised by students who stare at a problem without writing anything down for one, five, or even ten minutes, and then declare that they are giving up. Just as it is a big help to make a graph or a table, writing ideas down where you can see them and work with them provides most problem solvers with stronger leverage with those ideas than if they keep all those ideas in their head.

Example Find all the values of n that satisfy $n^2 = 11p + 4$ where p is a prime. After playing with this and getting nowhere, you decide to try rewriting this as $n^2 – 4 = 11p$. As soon as you see this on the page, it occurs to you that the left side is a difference of squares, and so you rewrite this again as $(n – 2)(n + 2) = 11p$. At this point, the problem is easy and you quickly come to the solution $n = 9$.

■ Work the problem both backwards and forwards and see if it meets in the middle. #### Fill this in.

For proving X is a Y, generate subgoals by looking at results that guarantee that something is a Y. For example, if Y is a parallelogram, then you have potential subgoals that are: the diagonals of X bisect each other; opposite sides of X are congruent; opposite angles of X are congruent.

■ Be persistent. Also, be willing to put the problem aside for a while, even overnight, and return to it later.

Be sure to apply as many of these problem-solving techniques on your problem as seem to be useful. Also, give your brain a chance to work some magic behind the scenes. It is amazing, and extremely rewarding, to put a problem down for a while and pick it up later and suddenly think of the solution that was eluding you.

If you are working on the right types of problems, you will not be able to solve every problem and that is okay. Embrace the challenge, have fun with it, and learn things from your successes and your unsuccesses (as Polya put it).

■ If you are having trouble proving that something is true, have you tried looking for counterexamples?

The search for counterexamples can be very instructive. Of course, if you find a counterexample, your work is done, and you can use your counterexample to show that the original problem is false. If you seem unable to find a counterexample, the problem requirements that keep your examples from being counterexamples may give you strong leads as to why the original problem is true, and that may be exactly what you need.

Example You are asked to prove or disprove that if you add 1 to the product of the first n prime numbers the result is always a prime. You try the first few cases and see that it seems to be true. You analyze the situation and see that of course none of the prime numbers involved in the product can divide the result, but you become suspicious that there is no reason some larger prime might not be a factor. Also, you remember that nobody has ever come up with a formula that always produces primes. So, you try more examples, and eventually find the counterexample 2 * 3 * 5 * 7 * 11 * 13 + 1 = 59 * 509.

Example If you remove the two opposite white corners of a chessboard, prove that it is impossible to exactly cover the resulting board with 2x1 dominoes. At first you think this must be wrong. You see that you have 62 squares to cover with a simple piece that has two squares in it, and you think this should be easy. As you try examples involving small boards, say 4x4 or 6x6, you quickly realize that you always end up with two extra black squares no matter what you do. You also notice that each time you place a domino you reduce the number of uncovered white squares by one and the number of uncovered black squares by one. These realizations lead to the proof of why it is impossible!

■ Assume the result or your proposed answer is true. What must be true for that to happen? There are three situations covered here. In geometry construction problems, you may be asked to construct a point, line, or figure - go ahead and draw the figure as part of your diagram and see what that suggests to you. If you are asked to prove that something is true, assume it is true and see what other statements would naturally follow or depend on that. Finally, you may be asked to find a result and you may have a possible solution - by assuming it is correct, you may see things that naturally follow from it or depend on it. If you create a world in which your result is true, you will often realize that there are other aspects of the problem space that must occur. These aspects can provide you with intermediate goals or with insights as to how the problem pieces must fit together.

Put in one or more examples.

For geometry construction problems, draw the figure and see what properties it must have.

■ Look at proof by contradiction or contrapositive - assume the conclusion of the problem is false. This can lead to at least two directions of attack. For some problems, the contrapositive of the original problem is easier to understand and attack. For other problems, you may decide that proof by contradiction is the easiest way to get a handle on the problem. The distinction between these two approaches may be so subtle as to be unimportant for some problems.

Example Prove that if n is composite, then n must have a prime factor less than or equal to its square root. The contrapositive of this is that if all of the primes factors of a number are greater than its square root, then the number must be a prime. This contrapositive may seem easier to prove.

Example Prove that for any 22 days, at least four of them fall on the same day of the week. Suppose the conclusion is false. That would mean that there are at most three days for each day of the week. Consequently, there could be at most 7*3 = 21 days involved. This contradicts the statement that there are 22 days.

Example Prove there are an infinite number of primes. Assume this is false. That would mean that there is a finite list of primes. You might quickly become suspicious that you can create a large number that would not be divisible by any of the primes on your list. At this point the end is in sight and it's just a matter of playing around.

When proving uniqueness, proof by contradiction is often helpful.

■ Exploit symmetries in the problem. #### finish writing this.

■ Look for extreme points in the problem that may force actions or relationships. Problems sometime have extreme values or locations that force other things to be true or to interact.

Example Suppose that the friendship relationship is reciprocal. Prove that in a group of n friends there must always be at least two people with the same number of friends in the group. If there were no duplication, then the n people would have to have distinct numbers of friends with values from 0 to n-1. The two extremes of 0 and n-1 are important – it is not possible for both to happen, and the problem solution follows

Consider extreme cases and special cases

Calculating (or approximating) values over a range of cases may suggest the nature of an extremum which, once "determined," may be justified in any variety of ways. Special cases of symmetric objects are often prime candidates for examination.

■ Use Geometry strategies. Here is a list of strategies to use when dealing with Geometry problems.

Finish writing this

■ Use Algebra II and Precalculus strategies. Here is a list of strategies to use when dealing with Algebra II and Precalculus problems.

Finish writing this

■ Use AP Calculus strategies. Here is a list of strategies to use when dealing with AP Calculus problems.

Finish writing this

You are done with the problem, now what? There are important ideas to consider once a problem solver has finished a problem. These ideas are often ignored by the student due to time pressure or the feeling that the job is done, and it is time to move on to the next thing.

■ Have you checked your answer and your solution steps? While it is always a good idea to look back over your steps, if there is a mistake in those steps it is often easy to have the same thought processes while reviewing the steps and thereby repeat the mistake and not catch it. One way to avoid this is to find alternate ways to do things. Here are some other ways of finding errors in your work.

a. Remember to use the steps from the Basic Strategies section.

b. Might the original problem have more solutions than just the ones you thought of?

c. If your problem produces a general result, try applying that result to specific cases and seeing if it produces correct conclusions.

Example If you have a general result about trapezoids, you should be able to set the length of one of the two parallel sides of the trapezoid to 0 and get a true statement about triangles.

d. Does your result work if you move it to a more general setting?

Example Finding the length of a diagonal of a rectangle involves using the Pythagorean theorem with a right triangle. That same idea will probably work and produce a similar formula when finding the length of the diagonal of a three-dimensional box.

■ Can you think of a different way to solve this problem?

Having a solution in hand often provides the confidence and new insights to see the problem in a new light and find different and sometimes cleaner, more satisfying solutions to the problem.

Example (Polya) A frustrum is a cone whose top has been sliced off parallel to the base. Asked for the lateral surface area of a frustrum of a right circular cone with lower radius R, upper radius r, and height h, you get $\pi(R+r)\sqrt{(R-r)^2+h^2}$. Looking at $\pi(R+r)$, you realize this is $(2\pi R+2\pi r)/2$, which is the circumference of the midsection of the frustrum. Looking at $\sqrt{(R-r)^2+h^2}$, you realize this is the slant height of the frustrum. Using these two insights, your formula becomes the circumference of the midsection times the slant height. This is a much cleaner and more intuitive formula.

■ Can you think of ways of generalizing or specializing the original problem that would be interesting to investigate? The problems given in your homework do not need to be the end of your math explorations!

As was discussed earlier in the problem-solving strategies, sometimes this produces a problem that is easier to solve. This is also how new mathematics gets created – the mathematician sees a result, wonders if it can be applied more widely, and then starts exploring for interesting new results.

Example You might be asked to show that the sum of the squares of two integers is always of the form 4k, 4k + 1, or 4k + 2. This may lead you to wonder about which numbers can be expressed as the sum of two squares. This new question is a rich playground with interesting results to look into, such as: Which primes are expressible as the sum of two squares? Why is it that if you take two numbers which are a sum of two squares then their product is also a sum of two squares?

Example You are asked to find the product of the divisors of a number. In doing this you discover that the formula depends on there being an odd number of divisors for numbers that are squares. This naturally leads to an investigation of why some numbers have an even number of divisors and others an odd number.

Advanced Strategies

- Look at simpler versions of the problem. Try lots of examples and look for patterns.
- Use parity e.g. (positive/negative, even/odd) to simplify the analysis
- For repeated actions, look for an invariant function that does not change
- Look for symmetry in the problem
- Use the pigeonhole principle (aka Dirichlet's box principle)
- Number Theory strategies
- Inequality strategies – squares are nonnegative, AM-GM inequality, Cauchy-Schwartz, Chebyshev
- Combinatorics strategies – see p. 87 of Engel's book
- Look for a maximal or extremal element or state
- Coloring proofs – partitioning a set into disjoint sets
- For processes built on earlier results, look at weak or strong induction

The following are detailed discussions of each of the problem strategies listed above.

■ Have you looked at simpler versions of the problem? Along the same lines, have you tried lots of examples and looked for patterns in the way the examples work out? Sometimes, just the process of working the examples illuminates what the process of the problem is and how to solve it.
Yes, this is repeated from the Basics section. This strategy has more basic versions and some sophisticated ones, and it is worth looking at all of them.
####
If there are a large number of variables in a problem, all of which play the same role, look at the analogous 1- or 2-variable problem. You may be able to build up a solution from there.
####
When dealing with problems that concern the roots of polynomials, it may be of use to look at easily factorable polynomials.
When dealing with problems that concern sequences or series that are constructed recursively, it may be of use to try initial values of 0 and 1 - if such choices don't destroy the generality of the processes under investigation.
2) For questions about roots of complex algebraic expressions, choose special cases where the roots or easy to keep track of (e.g. easily factored polynomials).
3) In iterated computations or recursions, choose easy starting points such as 0 and 1.

Number Theory strategy
For divisibility problems – consider using unique factorization
For divisibility problems or remainder problems – consider using modular arithmetic

Chapter 3 - Self-Monitoring

At any given moment as a person is solving a problem there are choices to be made about what to do and how long to do it. Some typical choices are: should I continue what I am doing, should I abandon this approach permanently or temporarily, should I switch to another approach, and should I do more analysis and discover another approach? The effectiveness of this self-monitoring and choice making can be the difference between success and failure.

In Schoenfeld's studies, roughly sixty percent of test subjects (upper high school or early college students working in pairs) in a 20-minute solution attempt read the problem, made a decision quickly as to what to do, and then pursued that solution approach no matter what happened for the remainder of the time. That first decision, never reconsidered, generally guaranteed failure for those test subjects. Good problem solving requires frequent self-monitoring and assessing of where things stand and what should be done next. It is essential that students develop the habit of brief periodic mental check-ins, on the order of once a minute.

The habits of self-monitoring and assessing are difficult to codify, but the following are some thoughts.

Play with the problem and be persistent. This is as much a belief as it is self-monitoring, but I think I like it here better. Be persistent and be willing to put the problem aside for a while and return to it later.

If you find yourself becoming overly frustrated or perhaps too tired to press on, take some time off to regroup. When you come back to your problem, review what you have done so far. In particular, look carefully to see whether you might have overlooked or misunderstood parts of the problem description. Also, just having some time away from the problem can be useful for letting a different part of your brain work in the background.

Some questions one might ask are:

> Should I pursue this line of thought further, or should I back off (put it aside by not throw it away) and consider another alternative?
> Should I take this opportunity to create a list of possible next steps by applying some problem-solving strategies that appear to apply to this situation?
> How do I rank the possible next steps I have generated? Generally speaking, quicker steps would rank higher because they can be dispensed with quickly. Complex calculations should generally be avoided unless no simpler options are available. However; a particularly promising long calculation might be preferred if confidence in it is high.

One research hypothesis put forward is that a person's internal dialog of self-monitoring for problem solving is developed during cooperative problem solving with a group. The frequent questioning and challenging of group members with each other for what is being done and considering what should be done next becomes a model for what the individual will do internally. To the extent that this hypothesis is correct, this points to a particular potent value of doing group work.

Schoenfeld describes how a teacher can promote the practice of self-monitoring. The point is to make the asking of these questions by the teacher habitual for the students for themselves. After giving the class a problem to work on, the teacher moves from group to group and is only allowed to ask one of three questions:
1) “What (exactly) are you doing? (Can you describe it precisely?)
2) Why are you doing what you are doing? (How does it fit into your vision of the solution?)
3) How does it help you? (What will you do with the outcome when you obtain it?)

Chapter 4 - Beliefs and Attitudes

A student's beliefs and attitudes shape and are shaped by the student's interactions with mathematical tasks. What follows is far from an exhaustive list of examples of how beliefs may influence how a person goes about approaching mathematics. While most of these beliefs are given in a negative version, many can be reversed to produce a positive version.

Over the last three decades, Carol Dweck's work on growth mindset has been a significant example of how a person's mindset can affect their performance. I will assume the reader is familiar with this work and not belabor what has been repeated quite often.

Overall math success and culture:

- (Parent speaking) It's okay to not be good at math. I was never very good at it either.
- People who are good at math are nerdy and social misfits.
- Mathematics work is a lonely activity, performed by a single individual.
- Women good at math will be viewed as lacking femininity and not fit in socially.
- (Fixed Mindset) You are either born with math talent or you are not.
- I am good (bad) at math, so my ideas are generally useful (useless).
- Weak students attribute their math successes to luck and their failures to lack of ability.
- Strong students attribute their successes to their abilities.
- The better the student is, the less likely he or she is to believe that mathematics is mostly memorizing, that success depends on memorization, or that problems get worked from the top down in step-by-step procedures.
- Those that see themselves as being good at mathematics also find it interesting.
- Better students perceive themselves as working harder in mathematics than most.
- US Parents believe that reading, not mathematics, needs more emphasis in the curriculum.
- In the US (more so than in Japan), people are much more likely to believe that innate ability (as opposed to effort) underlies children's success in mathematics

The relevance of math work:

- I will never use this mathematics in my real life.
- Formal mathematics has little to do with real thinking.

Problem solving:

- I believe that my efforts do (don't) have the potential of being successful.
- Only geniuses are capable of discovering or creating mathematics.
  - Consequence: If you forget something, do not try to recreate it. Accept procedures at face value and do not try to understand why they work.
- If I cannot solve a problem in 2 minutes (or 5 minutes), it must require an expert.
  - Consequence: Give up if a problem takes too long. Do not value or enjoy the process of problem solving.

(Teaching issue) Math is performing exercises. Most math tasks, especially test problems, are designed to be done in a few minutes.

Consequence: Students have little experience doing complicated problems over periods of time. Consequently, they have no expectation that progress can be made in that circumstance.

Immediately after reading a problem statement, I should understand what is asked.

Each step must be accurate without trial and error experiments.

Attaining the solution is the most important aspect of problem solving activity.

Problem solving is a linear process.

Hypotheses are tested as they come up. They are not ranked for likely effectiveness.

Naïve empiricism:

A proposition is true if it sounds true or rings true.

Consequence: There is no need to criticize or revise - intuitive feel suffices.

Proofs and deductive reasoning are needed when you already know or are told the answer and you need to prove it is correct.

Consequence: One does not need deductive reasoning to find an answer.

When I do a proof, I can only verify something a mathematician has shown to be true.

Formal mathematics has little to do with real thinking.

Consequence: In a problem that calls for discovery, formal mathematics is not used.

You have to memorize the way to do geometric constructions.

Consequence: Mathematical deduction is not a tool of invention.

Geometric insight comes from very accurate drawings.

Verification of steps is purely empirical. Constructions are verified by doing them.

Passive versus Active:

The math that I learn in school is mostly facts and procedures that have to be memorized.

It's okay to use mathematics that you do not understand as long as you know how to use it.

Mathematical theorems and practices are handed down by mathematicians and cannot be derived by students.

Learning mathematics is mostly memorizing.

Doing mathematics requires lots of practice in following rules.

There is always a rule to follow in solving mathematics problems

Doing mathematics means following the rules laid down by the teacher, knowing mathematics means remembering and applying the correct rule when the teacher asks a question, and mathematical truth is determined when the answer is ratified by the teacher.

Mathematics problems have one and only one right answer.

There is only one correct way to solve any mathematics problem - usually the rule the teacher has most recently demonstrated to the class.

Ordinary students cannot expect to understand mathematics; they expect simply to memorize it and apply what they have learned mechanically and without understanding.

Mathematics is only calculation, algorithm, and rule memorization.

Meaning versus Form:

A geometry proof must be written in an exact form that the teacher asks for.

“Being mathematical” means expressing oneself via the prescribed forms.