Contents

5 Number Theory 95

6 Fractions 126

1.4 Place values increase by factors of 10

Learn that the value of each place increases by a factor of 10.

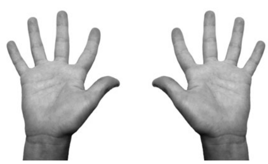
This is a key step in understanding and working with multi-digit numbers, so make sure it is very well understood before moving on.

Place value increases by 10 Place value refers to the value assigned to each place in a number. In the number 237, the 7 is in the 1's place, the 3 is in the 10's place, and the 2 is in the 100's place. The value of each place, first 1, then 10, then 100, increases by a factor of 10 as you move to the left in the number.

The term *factor* is used frequently throughout this book.

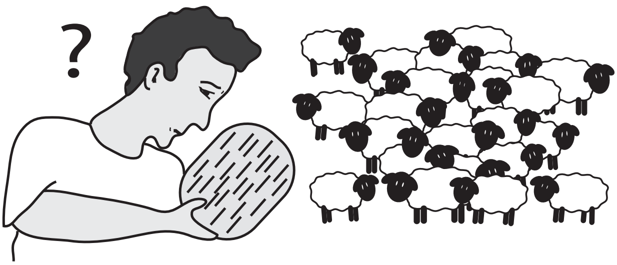
A number "increased by a *factor* of 10" has been multiplied by 10. A number has a *factor* of 5 if 5 evenly divides the number.

Where does the 10 come from? You would be amazed how few children in 7^{th} grade know that we use a factor of 10 for place value because we have 10 fingers.

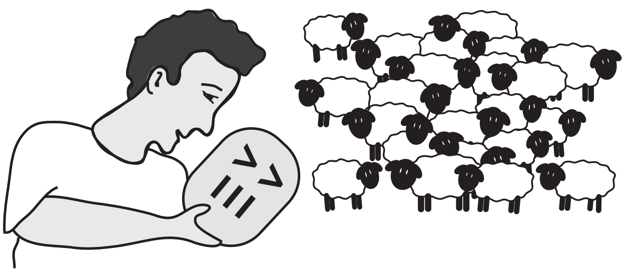


To explain the need for bundling, talk with your child about how a tradesman or shepherd ten thousand years ago would keep a record counting a collection of things.

A person making records a long time ago would put one tally mark down to record each item. However, it was not much easier to see how many things there were looking at the marks than it was looking at the original items. The answer was to start bundling.



Bundling To simplify things, people used a strategy of bundling. They would count things on their fingers, and each time they ran out of fingers they would mark that down as a bundle. When they were done, they would also mark down how many things were not in a bundle. For example, if they had 23 sheep, they would end up with 2 bundle-marks for the bundles of ten, and 3 marks for the sheep left over.



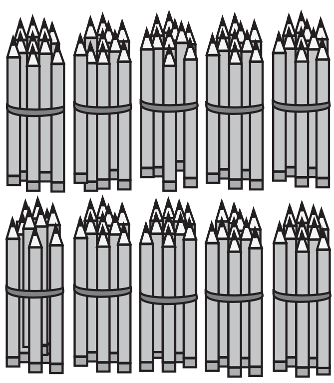
Practice this with your child with something you have lots of that you can make bundles with. Popsicle sticks, toothpicks, and Legos are examples of such things that can be rubber-banded or stuck together in bundles.

Put an assortment of the counting items on a table, and have your child make as many bundles of 10 as possible and write down the corresponding number. Sometimes, rather than using standard written numbers, have fun creating special marks to use for bundles and left-over individual items.

Bigger bundles This idea of bundling was used with even bigger numbers. For a number in the hundreds, there would be lots and lots of bundles of 10. Rather than having all of those bundles to keep track of, each time there were 10 bundles of 10, this would be made into a 100-super-bundle.

For even larger numbers, if there were lots of 100-super-bundles, these 100-super-bundles would be bundled in groups of 10 forming 1000-super-bundles. Make sure your child understands that the factor of 10 comes from the maximum number of things you can count on your fingers.

Practice bundling with your child by creating in advance a number of bundles of 10. Then, give your child a collection of bundles of 10 and of loose items, and have your child make bundles and super-bundles and write down the number. Keep emphasizing that each bigger size of bundle is made out of 10 of the smaller size items or bundles.



Written numbers When practicing bundling with your child, talk about the digits in the numbers that are produced. Do some examples where the numbers are repeated. For example, produce the answer 225. Talk about how the 2 in the hundreds place is worth 10 times as much as the 2 in the tens place. Explain that this is because each bundle of 100 is 10 groups of bundles of 10.

Talk with your child about how each place in a number is named after the bundle size it is counting. Take any number, such as 287. Talk about how the 7 is counting 1's, the 8 is counting bundles of 10, and the 2 is counting bundles of 100.

If you want, compare our system of using place value to some other systems described in Chapter 9: *Ancient Number Systems*. For example, talk about how the Romans used different symbols to indicate different bundle sizes—X to indicate a bundle of 10, and C to indicate a bundle of 100. Rather than creating different symbols for each different bundle size, we use place value and use different positions in the written number.

Practice regrouping After much practice with bundles, do the following practice trading off values in different number places. Write down labeled columns for the 1's, 10's, and 100's columns as shown below. Add in higher value columns once your child gets comfortable with the smaller values. Next, pick some number to practice with, such as 364 shown below.

$100's$	$10's$	$1's$
3	6	4
$2 + \underline{1}$	6	4
2	$\underline{10} + 6$	4
2	16	4
2	$14 + \underline{2}$	4
2	14	$\underline{20} + 4$
2	14	24
2	14	$\underline{10} + 14$
2	$14 + \underline{1}$	14
2	15	14

Show how you can trade 1 in one column for 10 in the column to its right. Also, go the other way, by showing how 10 in a column can be traded for a 1 in the column to its left. Emphasize that the total value of the number is never changing, that only the representation for it is different.

Seeing a group of things two ways The tricky part of this step for a child is being able to look at 10 individual things two ways. The 10 things need to be seen as both 10 separate things, and as one group of 10. This sounds easy to us, but it is a big developmental step that your child may take a while to be ready for. If your child has a lot of trouble with this now, do some other area of math for a while and try this again in a few months.

Compare with other bases It is very helpful to compare our use of base 10 with other number bases. The subject of other number bases is covered in Chapter 8: *Number Bases—Animal Math* and in Chapter 9: *Ancient Number Systems*. Seeing numbers represented in other bases makes the factor of 10 we use have a meaning, and makes it less taken for granted.

RELATED STEPS

The material in Section 3.1: *Powers of 10* should be done with, or just after, this section.

The bundling practice for this step is very similar to what you will be teaching your child about *expanded numbers* in the next section. The difference between the two is a difference in emphasis. In this section you are emphasizing the size of the bundles, and how that size increases by factors of 10. In Section 1.5: *Expanded number format* you will take the bundle sizes for granted, and the practice will be in breaking up the numbers into those size bundles.

2.10 Subtracting with regrouping

LESSON

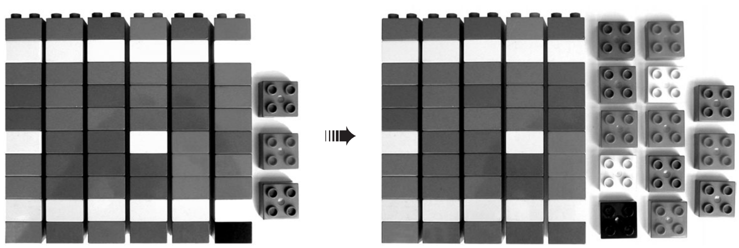
> Learning to subtract when regrouping is involved.

Practice

Bundles The same techniques you used for introducing subtraction without regrouping in Section 2.5: *Add & subtract without regrouping* will be useful here.

Start by going through some problems with bundles of popsicle sticks or Legos, showing how a bundle of 10 sticks can be split apart and used as 10 single sticks when needed.

Here is 63 being regrouped as 50 + 13.



Expanded numbers Start writing out problems using numbers written out in expanded number format.

$$\begin{array}{r} 63 \\ -\ 25 \\ \hline \end{array} \quad \Rightarrow \quad \begin{array}{r} 60 + 3 \\ -\ 20 + 5 \\ \hline \end{array} \quad \Rightarrow \quad \begin{array}{r} 50 + 13 \\ -\ 20 + \ \ 5 \\ \hline 30 + \ \ 8 \end{array}$$

Marking regrouping There is one main method used to mark regrouping for subtraction, so you will not have multiple methods to choose from, the way there was for addition. You may enjoy playing with some of the unusual methods used for regrouping collected together in Section 2.12: *Unusual subtraction regrouping methods*.

To mark changes for a regrouping, put a single line through any number being reduced and write its new value above it. This cross-out mark should be done lightly enough that the original number is still easy to read. For any group of ten being regrouped, put a smaller 1 on the left of, and slightly above, the digit that is to receive the regrouping. For example:

$$\begin{array}{r} 63 \\ -\ 25 \\ \hline \end{array} \quad \Rightarrow \quad \begin{array}{r} \overset{5}{\cancel{6}}\,{}^{1}3 \\ -\ 2\ 5 \\ \hline \end{array}$$

Method to avoid larger subtractions This is a way to do subtractions so that your child never needs to subtract from a number larger than 10. I am not an advocate of this method, since it makes regrouping a bit more complex. However, if you or your child prefer this method, feel free to use it.

Look again at the 63 − 25 example. After regrouping the 1, immediately subtract 5 from the regrouped 10 ones, and add that result to the 3. That is, calculate

$$(10 - 5) + 3 = 5 + 3 = 8$$

rather than

$$(10 + 3) - 5 = 13 - 5 = 8$$

By subtracting from the regrouped 10 first, before it has been combined with the number that needed it (3 in this case), you never have to subtract from numbers larger than 10.

In the first book, I described doing problems such as 13 − 5 by adding two differences

$$(13 - 10) + (10 - 5)$$

Using the two differences is equivalent to the method given above for subtracting from regroupings. The reason I do not like this method for regrouping is that I feel your child will have internalized how to do 13 − 5 before reaching this step, and I prefer to keep the steps for regrouping as simple as possible.

2.11 Subtraction regrouping—step progression

LESSON

The problem progression to teach regrouping for subtraction.

Practice

Practice regrouping for subtraction by following these steps. As with addition, each of these steps may take anywhere from a few minutes to a few weeks. Be flexible and patient, and remember that there is no schedule for this.

Whenever regrouping, be sure to emphasize place value. It is very important that the process of regrouping 1 from one column to produce 10 in the next column is understood, and is not just mechanical.

Step 1: Two-digit numbers Start with simple problems that involve two-digit numbers, where the regrouping is needed in the ones column. Give some problems with a 0 in the ones digit in the top number, and see whether that is handled smoothly or not.

$$\begin{array}{r} 54 \\ -\ 39 \\ \hline \end{array} \qquad \begin{array}{r} 92 \\ -\ 46 \\ \hline \end{array} \qquad \begin{array}{r} 80 \\ -\ 27 \\ \hline \end{array}$$

Step 2: Single regrouping in other columns Continue to have a single regrouping, but now increase the size of the numbers and vary the column where the regrouping occurs. Mix in examples where the bottom number has fewer columns than the top number.

$$\begin{array}{r} 258 \\ -\ 174 \\ \hline \end{array} \qquad \begin{array}{r} 592 \\ -\ \ \ 46 \\ \hline \end{array} \qquad \begin{array}{r} 853 \\ -\ 393 \\ \hline \end{array}$$

Step 3: Two separate regroupings Use 4-digit subtraction problems with regroupings that occur in the first and third columns. Since the regroupings are in separated columns, there should be nothing tricky about this, but it is good to practice and check.

$$\begin{array}{r} 6{,}754 \\ -\ 2{,}839 \\ \hline \end{array} \qquad \begin{array}{r} 3{,}092 \\ -\ \ \ \ 446 \\ \hline \end{array}$$

Step 4: Two regroupings together The problems for this step have regrouping in two neighboring columns. Use three- or four-digit numbers for these problems.

$$\begin{array}{r} 1{,}254 \\ -\quad 839 \\ \hline \end{array} \qquad\qquad \begin{array}{r} 2{,}892 \\ -\ 1{,}396 \\ \hline \end{array}$$

The regrouping marks can get a little messy, so an example done in stages is given below:

$$\begin{array}{r} 526 \\ -\ 249 \\ \hline \end{array} \quad \Rightarrow \quad \begin{array}{rrrr} & & {\scriptstyle 1} & \\ & 5 & \cancel{2} & {}^{1}6 \\ - & 2 & 4 & 9 \\ \hline & & & 7 \end{array} \quad \Rightarrow \quad \begin{array}{rrrr} & {\scriptstyle 4} & {\scriptstyle 11} & \\ & \cancel{5} & \cancel{2} & {}^{1}6 \\ - & 2 & 4 & 9 \\ \hline & 2 & 2 & 7 \end{array}$$

Step 5: Regroup from a 1 in left column These problems involve regrouping from a 1 in the left-most column. Check that your child is comfortable crossing out the 1 and putting a 0 there, and then having a blank (or 0) in that column in the result. If it's fun putting a 0 in the left-most column of the result, let your child have fun.

$$\begin{array}{r} 14 \\ -\quad 9 \\ \hline \end{array} \qquad\qquad \begin{array}{r} 124 \\ -\quad 46 \\ \hline \end{array} \qquad\qquad \begin{array}{r} 1{,}633 \\ -\quad 807 \\ \hline \end{array}$$

Step 6: Single regrouping from 0 Give problems involving a single regrouping next to a 0.

$$\begin{array}{r} 702 \\ -\ 195 \\ \hline \end{array} \qquad\qquad \begin{array}{r} 2{,}902 \\ -\ 1{,}486 \\ \hline \end{array}$$

This is tricky the first couple of times, because the regrouping from the 0 first requires a regrouping from the digit to the left of the 0.

An example done in stages is shown below:

$$\begin{array}{r} 504 \\ -\ 359 \\ \hline \end{array} \quad \Rightarrow \quad \begin{array}{r} \overset{4}{\cancel{5}}{}^{1}0\ 4 \\ -\ 3\ 5\ 9 \\ \hline \end{array} \quad \Rightarrow \quad \begin{array}{r} \overset{4}{\cancel{5}}{}^{1}\overset{9}{\cancel{0}}{}^{1}4 \\ -\ 3\ 5\ 9 \\ \hline 1\ 4\ 5 \end{array}$$

In this last example, it is important to regroup 1 from the 5 first. If the 0 is crossed out and replaced by a 9 first, a child can get distracted and forget to do the other half of the regrouping. By crossing out the 5 first, there will be no problem picking up where your child left off.

Step 7: Regrouping across several 0's Extend the exercises from step 6 by giving problems that involve a single regrouping next to two or more consecutive 0's.

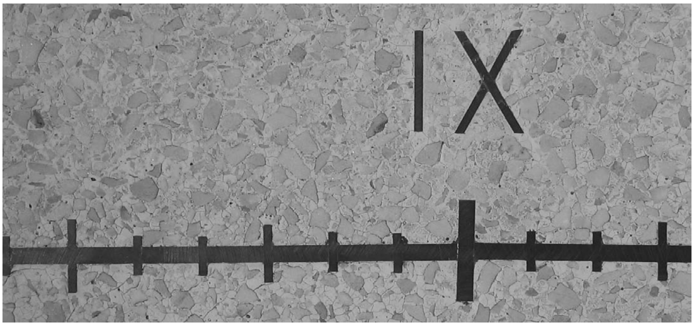
$$\begin{array}{r} 5{,}004 \\ -\ 3{,}259 \\ \hline \end{array} \qquad\qquad \begin{array}{r} 90{,}002 \\ -\ 40{,}236 \\ \hline \end{array}$$

As in step 6, the regrouping will ripple to the left until you reach the first non-zero number to regroup from. As before, it is very important to start marking the regrouping next to the left-most 0 and work back to the right. For example:

$$\begin{array}{r} 7{,}002 \\ -\ 1{,}357 \\ \hline \end{array} \quad \Rightarrow \quad \begin{array}{r} \overset{6}{\cancel{7}}{}^{1}0\ 0\ 2 \\ -\ 1\ 3\ 5\ 7 \\ \hline \end{array} \quad \Rightarrow \quad \begin{array}{r} \overset{6}{\cancel{7}}{}^{1}\overset{9}{\cancel{0}}{}^{1}0\ 2 \\ -\ 1\ 3\ 5\ 7 \\ \hline \end{array} \quad \Rightarrow$$

$$\begin{array}{r} \overset{6}{\cancel{7}}{}^{1}\overset{9}{\cancel{0}}{}^{1}\overset{9}{\cancel{0}}{}^{1}2 \\ -\ 1\ 3\ 5\ 7 \\ \hline 5\ 6\ 4\ 5 \end{array}$$

Step 8: Mix together all types Finally, mix up all of the preceding types. Emphasize any particular types that your child is having trouble with.



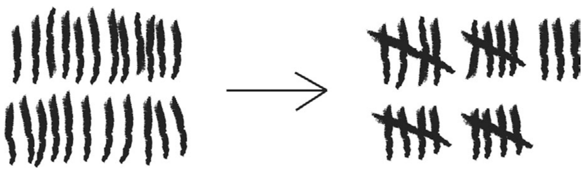
CHAPTER 9 Ancient Number Systems

This chapter is entirely for fun. This material ties into the ideas of place value and number bases. It also gives an historical perspective on the development of number systems from ancient times. Learning the Roman numerals is the only traditional portion of this chapter, and the rest can be skipped if there is no interest in it.

Tally marks to symbols When people first started recording quantities, they did so using a series of marks on a rock or a piece of wood. The person would put down one mark for each item to be counted. This works well until there are 20 or more marks, at which point it becomes difficult to size up the number from looking at a series of ungrouped marks.

The oldest artifacts of this type are around 30,000 years old. Interestingly, this is also about the time that cave art seems to have started. At about the time writing systems started developing 5,000 years ago, more sophisticated ways of recording quantities started to be used.

An early improvement was to put the marks in groups. People still do this today when counting things and recording them as they go. A person will put down a series of marks, until it is time for the fifth mark. At that point, a cross mark is made to indicate the fifth item, and to make it easier to see the group of five. Since five things are what a person can count on one hand, groups of five are natural for us.



Some of the systems borrowed number ideas from societies they traded with, and some invented their systems entirely on their own. Given all of the progress that occurred in so many areas over the last 5,000 years, it is remarkable how slowly reckoning skills developed.

Historical key steps There is a small set of key turning points in the development of number systems.

1. **Make symbols** To avoid have a long list of marks or 1's, systems created special symbols for some of the quantities. This is the starting point for all of the number systems.
2. **Add, subtract, multiply** In order to avoid making a huge number of symbols, they had ways to combine the symbols. This often involved being able to add two or more symbols together. For example, the Romans formed 11 by putting the symbol for 10 together with the symbol for 1.

   Some versions of the Roman system also used subtraction to form numerals. If the symbol for 1 was placed before the symbol for 10, then the value was $10 - 1 = 9$.

   The Chinese used multiplication to combine the symbols for 1 through 9, with their symbols for powers of 10. For example, 80 was formed by putting the symbol for 8 before the symbol for 10. The Roman and Greek systems multiplied the value of a group of symbols by 1,000 by putting a mark before them or over them.

3. **Symbols for powers of the base** Special symbols were created for values that were powers of the base. For example, the only symbols that the Egyptians had in the hieroglyphic system were for the numbers 1, 10, 100, and so on. The Egyptians formed all other numbers by adding these values together.

4. **Place value** This was a major advance in number representations, as it made doing arithmetic far simpler. It also greatly reduced the need for special symbols.

   The only systems to incorporate the ideas of place value were the Babylonian, Mayan, Chinese counting board, and Hindu-Arabic. The Babylonians used base 60, the Mayans used base 20, and the other two used base 10.

5. **Use 0 with place value** The earliest place value systems used a gap or dot to indicate when a place had nothing in it. It was a significant development that the Hindu-Arabic and Mayan systems used 0's in their place value systems.

Ways to practice This chapter covers all of the major number systems developed in the ancient world. Most of these systems incorporate ideas about number bases. The first few sections of Chapter 8: *Number Bases—Animal Math* are good preparation for this. It works well to do those sections slightly before or at the same time as the examples in this chapter.

One way to play with these ideas is to pretend to be people in the different societies, and send each other messages using your numbers that the other person has to translate. You can also have fun writing secret "coded" messages to each other using numbers written in these various systems.

Go through the historical context of these various systems. Consider what was available before each system was created, what it was like to use the system, and look at the systems that came after them. Examples of such topics are: "Why was using numerals an improvement over using tally marks?", "What was natural about using base 5, 10, 20, and 60?", "How bulky and hard to read are Egyptian numbers compared to Chinese?", "How hard would it have been to learn the multiplication table using the Greek system?", "Why was using place value a natural follow on to using counting boards?", and "Why was it so hard for Europeans to give up the Roman numeral system in the Middle Ages?"

To get a better sense of the advantages and disadvantages of these systems, have your child do multi-digit addition, subtraction, and multiplication using them. Choose a few arithmetic problems, have your child do those problems using all of the systems, and discuss the differences.

More history If your child finds historical ideas interesting, there are a number of mathematical topics to look into to see how various societies dealt with them. For example, there were some interesting methods used for fractions in these early societies. The Egyptians used only fractions of the form $\frac{1}{n}$, except for the fraction $\frac{2}{3}$. The history of early geometry involves the Greeks, but also important developments in other societies, and your child may enjoy looking at how that got started. Some other important concepts that ancient societies struggled with were negative numbers and irrational numbers (such as $\sqrt{2}$ and π).

9.1 Egyptian

LESSON

Learning the ancient Egyptian number system.

Practice

The unified Egyptian civilization started around 3000 BC, and had a hieroglyphic form of writing at that time. As might be expected of a civilization that lasted thousands of years, their writing system, and their representation of numbers in particular, changed over time. The hieroglyphic form transitioned significantly to the hieratic form. The hieratic form later transitioned less dramatically to the demotic form.

All of these forms of writing numerals used a base 10 system for representing whole numbers greater than 0. They did not have a way to represent 0.

Hieroglyphs The Egyptians had hieroglyphs for each of the powers of 10, up to 10^6.

1	10	100	10^3	10^4	10^5	10^6

In order, these hieroglyphs are pictures of: single stroke, heel bone, coil of rope, water lily or lotus, finger, tadpole or frog, and man with both hands raised.

They formed numbers from these symbols by adding them together, using each symbol as many times as was needed. For example, to write the numbers 1,306 and 1,358,031 they wrote:

1,306 1,358,031

Hieratic The Egyptians used this system of writing after the invention of writing on papyrus. Initially the hieratic symbols looked very similar to the corresponding hieroglyphs, but over time the hieratic symbols diverged. Hieroglyphs continued to be used for symbols carved into stone.

Hieratic symbols use a simpler, more compact representation for numbers. However, there are a lot more symbols to learn.

This system uses a single symbol for each of the numbers 1, 2, . . . , 9, 10, 20, . . . 90, 100, 200, and so on. The symbols changed over the many hundreds of years they were used. Here is a sample from around 1600 BC.

1		10		100		1,000	
2		20		200		2,000	
3		30		300		3,000	
4		40		400		4,000	
5		50		500		5,000	
6		60		600		6,000	
7		70		700		7,000	
8		80		800		8,000	
9		90		900		9,000	

Writing 79 in hieratics requires just two symbols—a 70 and a 9. Hieroglyphs use 16 symbols—seven 10's and nine 1's.

The Greeks used a system of writing numbers very similar to the hieratic and demotic systems used by the Egyptians.